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"Anomalous" Refractive Index Dispersion Curves—A Relativistic Interpretation

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The Einstein field equations ($R_{\mu\sigma}=0$) are seen as possible candidates for a set of unified field equations. Three solutions of these field equations are used for a new interpretation and reformulation of the refractive index of an isotropic material medium. The new formulation explains the basic features of "anomalous" refractive index dispersion curves. It also predicts that the refractive index is a function of the angle of incidence when the plane in which the measurement is made is not tangential to the surface of the spherical gravitating mass, thereby providing a suitable test for the theory and hence of general relativity.

1. INTRODUCTION

The relationship between the refractive index of a material medium and the free space wavelength of electromagnetic radiation (henceforth EMR) has long been a subject of study (Cauchy, 1830; Koch, 1909; Born and Wolf, 1959; Wahlstrom, 1969). This work is limited to isotropic media where the refractive index is independent of direction or position in the medium when the effect of any gravitational field present is not included. In the passage of EMR through a material medium it is generally accepted that its velocity changes. This velocity change is accompanied by a wavelength change, the frequency remaining constant. Classically (Wahlstrom, 1969; Gall, 1979), this results in

$$n_c=\lambda/\lambda_1 \tag{1}$$

where n_c is the classical refractive index for the medium at the particular free space value of the wavelength (λ), and λ_1 is the corresponding

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wavelength in the medium. This classical formulation of the refractive index suggests a direct linear dependence of n on λ, which is not in agreement with the generally observed wavelength dependence (Cauchy, 1830; Koch, 1909; Born and Wolf, 1959; Wahlstrom, 1969). Thus various empirical relationships have been derived to try to express the actually observed relationship. Examples are the Cauchy equations (Cauchy, 1830; Born and Wolf, 1959; Wahlstrom, 1969):

$$n = 1 + a + (b/\lambda^2) + (c/\lambda^4) + \cdots \tag{2a}$$

or

$$n = A + (B/\lambda^2) + (C/\lambda^4) + \cdots \tag{2b}$$

where a, b, c, A, B, and C are constants. These equations suggest an inverse square dependence of n on λ, although higher-order terms $(1/\lambda^4)$ are also present. These equations are generally used in the visible spectral region for substances that do not absorb strongly. There is also the Koch equation (Koch, 1909) derived for gases (hydrogen, oxygen, and air) in the visible region of the spectrum:

$$n^2 - 1 = a_1 + b_1/(\lambda^2 - \lambda_0^2) \tag{3}$$

where a_1, b_1, and λ_0 are constants.

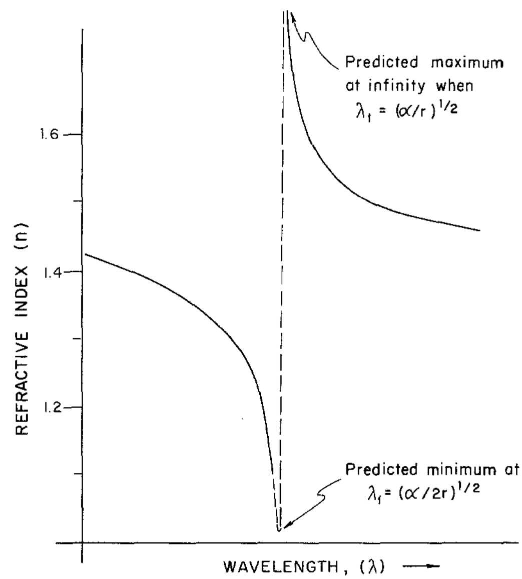

Fig. 1. Anomalous refractive index dispersion curve. Solid line: generally observed experimental curve; dashed-line curve: predicted behavior for $n_r^{\theta,\phi}$ in the inflection region.

Indeed the refractive index does generally decrease as the wavelength is increased in the visible region of the spectrum. In the region of an absorption band, however, an increase with the wavelength is also seen. The characteristic dependence in this region is referred to as "anomalous" refractive index dispersion curves (Born and Wolf, 1959; Wahlstrom, 1969; Kagarise, 1960), shown in Figure 1. The behavior at the inflection points (Kagarise, 1960) is often not shown (when shown it is generally inferred) because of the difficulty of measuring the refractive index in this region. Any theoretical explanation for these curves must be able to account both for an inverse and a direct dependence of n on λ as well as for inflection points where the dependence changes direction.

Mention is also made here of a related experiment (for reasons that will become obvious later in the paper) of current interest. In measuring the relativistic bending (Fomalont and Sramek, 1975) or the excess time delay (Shapiro et al., 1971) of radio waves in the gravitational field of the sun a direct square dependence on λ has generally been observed. This has been attributed to refraction by the solar corona.

2. THEORETICAL BACKGROUND

In a recent series of papers (C. A. Gall, 1979; C. Gall and O. Gall, 1979; C. A. Gall, 1979) a new point of view has been introduced on this phenomenon of refraction of EMR by material media based on general relativity. It was first necessary to include the effect of a material medium in the metric coefficients of the line element for an isotropic medium (C. A. Gall, 1979) in the absence of a gravitational field. The same criterion was applied as for the gravitational field, which is that the line element should satisfy the Einstein field equations ($R_{\mu\sigma}=0$). Essentially these field equations can then be considered as a set of unified field equations since the material medium and the gravitational field are now treated on the same basis (Adler et al., 1975; Einstein, 1953). Three solutions (C. A. Gall, 1979) were suggested, two of which gave the same classical dependence for the refractive index [equation (1)]. The third solution

$$ds^2=(\lambda_1^2/\lambda^2)c^2dt^2-(\lambda^2/\lambda_1^2)\left[dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \tag{4}$$

gave a new dependence for the refractive index:

$$n_r=\lambda^2/\lambda_1^2 \tag{5}$$

where n_r is now the relativistic refractive index. Since this third solution gives a square dependence on λ it is the more interesting result.

The fact that the velocity of EMR changes in a gravitational field implies that its wavelength must also change. This concept led to an alternative solution (C. Gall and O. Gall, 1979) of the field equations ($R_{\mu\sigma}=0$) for free space in the presence of a spherically symmetric gravitational field:

$$ds^2=\left[1-(\alpha/\lambda^2 r)\right]c^2dt^2-dr^2/\left[1-(\alpha/\lambda^2 r)\right]-r^2(d\theta^2+\sin^2\theta d\phi^2) \quad (6)$$

This result is interesting because of the inverse λ^2 dependence of the metric coefficient of the time coordinate.

In a third paper (C. A. Gall, 1979), a solution was obtained for these field equations ($R_{\mu\sigma}=0$) for an isotropic material medium in the presence of a spherically symmetric gravitational field:

$$ds^2=(\lambda_1^2/\lambda^2)\left[1-(\alpha/\lambda_1^2 r)\right]c^2dt^2-(\lambda^2/\lambda_1^2)$$

$$\left\{dr^2/\left[1-(\alpha/\lambda_1^2 r)\right]+r^2(d\theta^2+\sin^2\theta d\phi^2)\right\} \quad (7)$$

This third solution is essentially a combination of the previous two. It shows that when one measures the refractive index of a material medium, it is essentially impossible to separate from it in a purely additive way the relativistic bending (Fomalont and Sramek, 1975) due to any gravitational field present. [The converse operation (Fomalont and Sramek, 1975) is also true.] This is a direct consequence of the nonlinear nature (Adler et al., 1975) of the field equations ($R_{\mu\sigma}=0$). The generally accepted formulation of the refractive index [equation (1)] is essentially prerelativistic and as such could not have recognized this problem. It is therefore necessary to seek a new formulation. This problem will become even more obvious with the advent of experiments in orbiting laboratories or on other celestial bodies (such as the moon or mars) with gravitational fields and atmospheres different from those at the surface of the earth.

3. POSSIBLE FORMULATIONS

(A) A measurement of the refractive index generally involves very small distances so that the coordinate velocity of the EMR can be used. From equation (7) one obtains

$$r(d\theta/dt)=r\sin\theta(d\phi/dt)=c(\lambda_1^2/\lambda^2)\left[1-(\alpha/\lambda_1^2 r)\right]^{1/2} \quad (8a)$$

$$(dr/dt)=c(\lambda_1^2/\lambda^2)\left[1-(\alpha/\lambda_1^2 r)\right] \quad (8b)$$

The first idea that possibly comes to mind for defining the refractive index is to compare the coordinate velocity with the velocity of EMR in free space in the absence of a gravitational field (i.e. c). This gives

$$n_r^{\theta,\phi}=\lambda^2/\left\{\lambda_1^2\left[1-(\alpha/\lambda_1^2 r)\right]^{1/2}\right\}=\lambda^2/\left[\lambda_1^4-(\alpha\lambda_1^2/r)\right]^{1/2} \tag{9a}$$

$$n_r^r=\lambda^2/\left[\lambda_1^2-(\alpha/r)\right] \tag{9b}$$

The superscripts θ and ϕ indicate that the refractive index is measured in a plane tangential to the spherical gravitating body, while r indicates a measurement in the radial direction. This in itself is an important result since it means that the refractive index depends on the plane in which it is measured. Previous experiments could not have been expected to take this into consideration. The tangential plane is probably the most convenient to define and to carry out a measurement of the refractive index by means of angles. It is therefore interesting to examine the functional form of the expression (9a):

$$n_r^{\theta,\phi}=f(\lambda,\lambda_1,r) \tag{10a}$$

$$\lambda_1=f(\lambda) \tag{10b}$$

It should be noted of course that $n_r^{\theta,\phi}$ also depends on α, which is determined by the particular gravitating body, and that λ_1 depends on the particular medium. For points at the surface of the spherical gravitating body ($r=\text{const}$)

$$(dn_r^{\theta,\phi}/d\lambda_1)=\left\{2\lambda/\left[\lambda_1^4-(\alpha\lambda_1^2/r)\right]^{1/2}\right\}(d\lambda/d\lambda_1)$$
$$-\lambda^2\left[4\lambda_1^3-(2\alpha\lambda_1/r)\right]/\left\{2\left[\lambda_1^4-(\alpha\lambda_1^2/r)\right]^{3/2}\right\} \tag{11}$$

At an inflection point $(dn_r^{\theta,\phi}/d\lambda_1)=0$; so that

$$(d\lambda/d\lambda_1)=\lambda\left[2\lambda_1-(\alpha/\lambda_1 r)\right]/\left\{2\left[\lambda_1^2-(\alpha/r)\right]\right\} \tag{12}$$

At such a point also $(d\lambda/d\lambda_1)=0$; so that

$$\lambda_1=(\alpha/2r)^{1/2} \tag{13}$$

where the positive root was chosen since $\lambda_1>0$. The derivative of $n_r^{\theta,\phi}$ can also be calculated with respect to λ and the result equated to zero to obtain another condition for an inflection point. However, $(d\lambda_1/d\lambda)$ can be more simply obtained:

$$(d\lambda_1/d\lambda)=1/(d\lambda/d\lambda_1)=2[\lambda_1^2-(\alpha/r)]/\{\lambda[2\lambda_1-(\alpha/\lambda_1 r)]\} \quad (12a)$$

It is also to be expected that at this inflection point $(d\lambda_1/d\lambda)=0$, so that

$$\lambda_1=(\alpha/r)^{1/2} \quad (14)$$

These two results suggest that λ_1 varies between $(\alpha/2r)^{1/2}$ (its minimum value) and $(\alpha/r)^{1/2}$ (its maximum value). However, if equation (13) is substituted into (9a) an imaginary value is obtained for the refractive index. This suggests that this formulation of the refractive index should be reexamined.

(B) In an angle-type measurement of the refractive index one compares the angles of incidence (reference medium) and refraction (sample medium). However, the gravitational field present also affects the reference medium. This suggests defining the refractive index by comparison with the velocity of EMR in free space in the presence of the gravitational field [equation (6)]. This leads to

$$n_r^{\theta,\phi}=(\lambda^2/\lambda_1^2)\{[1-(\alpha/\lambda^2 r)]/[1-(\alpha/\lambda_1^2 r)]\}^{1/2} \quad (15a)$$

$$n_r^r=[\lambda^2-(\alpha/r)]/[\lambda_1^2-(\alpha/r)] \quad (15b)$$

Differentiating the tangential value of the refractive index [equation (15a)] gives

$$\begin{aligned}(dn_r^{\theta,\phi}/d\lambda_1)=(d\lambda/d\lambda_1)&[2\lambda^3-(\alpha\lambda/r)]\\ &/\{[\lambda^4-(\alpha\lambda^2/r)][\lambda_1^4-(\alpha\lambda_1^2/r)]^{1/2}\}\\ &-[2\lambda_1^3-(\alpha\lambda_1/r)]/[\lambda_1^4-(\alpha\lambda_1^2/r)]^{3/2} \quad (16)\end{aligned}$$

An inflection point is again obtained by putting $(dn_r^{\theta,\phi}/d\lambda_1)=0$ and $(d\lambda/d\lambda_1)=0$. Thus

$$(d\lambda/d\lambda_1)=[\lambda^3-(\alpha\lambda/r)][2\lambda_1^2-(\alpha/r)]/\{[\lambda_1^3-(\alpha\lambda_1/r)][2\lambda^2-(\alpha/r)]\} \quad (17)$$

Putting equation (17) equal to zero gives $\lambda_1=(\alpha/2r)^{1/2}$ which is the same result as equation (13). By considering $(dn_r^{\theta,\phi}/d\lambda)$ another inflection point is found at $\lambda_1 = (\alpha/r)^{1/2}$ which is the same result as equation (14). Remembering that $\lambda \geqslant \lambda_1$ and substituting (13) into (15a) gives a real value for the refractive index provided that $\lambda < (\alpha/r)^{1/2}$. Substituting (14) into (15a) gives an infinite value for the refractive index at the other inflection point. This then is an acceptable formulation and meets the criteria originally set. There are regions where $n_r^{\theta,\phi}$ increases and others where it decreases with respect to λ. [The dependence also approximates the observed (Cauchy, 1830; Koch, 1909; Born and Wolf, 1959; Wahlstrom, 1969) square law from the form of (15a).] And there are inflection points where it is a minimum [when $\lambda_1 = (\alpha/2r)^{1/2}$] and others where it reaches the maximum value of infinity [when $\lambda_1 = (\alpha/r)^{1/2}$]. This is all in agreement with the functional form of Figure 1.

(C) The formulation of the refractive index in Section B is ideally suited for a gravitating body like the moon which is devoid of an atmosphere. It can be applied on the earth but this requires the use of an artificial vacuum. Past work has for obvious reasons generally used the surface atmosphere (air) as the reference medium. It is therefore interesting to reformulate the refractive index by comparison with the velocity of EMR in such a reference medium (which varies from planet to planet) and in the presence of the gravitational field. This gives

$$n_r^{\theta,\phi} = (\lambda_a^2/\lambda_1^2)\{[1-(\alpha/\lambda_a^2 r)]/[1-(\alpha/\lambda_1^2 r)]\}^{1/2} \quad (18a)$$

$$n_r^r = [\lambda_a^2 - (\alpha/r)]/[\lambda_1^2 - (\alpha/r)] \quad (18b)$$

These equations are essentially similar to (15a) and (15b) with λ_a (the wavelength in the atmosphere) replacing λ. The only problem is that one has to remember that λ_a is not necessarily greater than λ_1 in all regions of the spectrum. Should this occur ($\lambda_a < \lambda_1$) the formulation breaks down as the refractive index is then less than unity. This has generally been recognized by experimenters who use nitrogen in the uv region (where oxygen absorbs) and then switch to an artificial vacuum further in the uv where nitrogen also absorbs—the so called vacuum uv.

4. CONCLUSION

The original relativistic refractive index defined in equation (5) is now seen in retrospect as not including the effect of the gravitational field. To some extent this behavior is approached in an orbiting laboratory (in free fall) since α is now determined by the mass of the laboratory, which is usually much less than that of the planetary body. It should be remembered, however, that at very small free space wavelengths (λ) even a small value of α, coupled with small values of r, can have an appreciable effect. The α factor should therefore always be included in the general formulation.

In past measurements of the relativistic bending of EMR (and related experiments) in the gravitational field of the sun, an important factor has been the refractive contribution (Fomalont and Sramek, 1975; Shapiro et al., 1971) of the solar corona. This analysis has been done from the opposite point of view, in which refraction by a medium is affected by gravitational bending. Indeed there is really no essential difference between the two types of measurements if one accepts the present approach, which leads to wavelength-dependent solutions of the field equations ($R_{\mu\sigma}=0$). The study of the bending in the gravitational field of the sun, however, is much more complicated since it involves (among other problems) large distances so that the coordinate velocity (which depends on the coordinate time) is no longer satisfactory. Measuring the refractive index, on the other hand, is a much simpler experiment and much easier to interpret.

One suggestion for testing the present theory and consequently general relativity involves measuring the refractive index by means of direct measurements of the angles of incidence and refraction. When the plane in which the measurement is carried out coincides with the tangential plane, then the refractive index should be independent of angle. If, however, the plane of measurement has a radial component then there should be an angular dependence. The measurements should be made in regions where there are large changes of n with λ. Indeed previous studies have indicated just such an angular dependence for concentrated solutions (Wahlstrom, 1969), but such studies did not specify the plane in which the measurements were done. A systematic study of this problem would be of great interest.

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